View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF COMBINATORIAL THEORY, Series A 37, 99–120 (1984)

Collineations of Projective Planes of Order 9

R. SHULL

Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02181

Communicated by the Managing Editors

Received June 17, 1982

Let P be a projective plane of order 9 other than one of the four known ones. Then the order of the full collineation group of P is $3^a \cdot 5^b$, where a is a nonnegative integer and $b = 0$ or 1.

1. INTRODUCTION

Associated with every finite projective plane $\mathscr{P}$ is a unique integer, called the *order* of $\mathscr{P}$, defined to be the number of points on any line of $\mathscr{P}$ minus 1. The order of a plane is well defined and equals one less than the number of lines through any point. For $n = 2$, 3, or 4 it is a relatively simple matter to show that there is only one plane of order n. A nice proof of the uniqueness of the plane of order 5 involving the characterization of projective planes as complete sets of mutually orthogonal Latin squares is given in [2]. There are no planes of order 6 by the Bruck–Ryser theorem. The planes of orders 7 and 8 have been shown unique through the use of long computer searches [3, 4]. To date, no noncomputer-aided proof has been given to establish either of these two results. Nine is the smallest order for which there is more than one plane. There are at least four projective planes of order 9: a Desarguesian plane, two Hall planes, and a Hughes plane. It is not known whether any other planes of order 9 exist.

Construction of additional planes of order 9 might well be attempted through a study of their possible collineation groups. That is the approach of this paper. The basic idea is as follows. We make an assumption concerning the nature of the collineation group C associated with a plane of order 9. In particular, we assume that some prime or prime power divides the order of C. This restricts the nature of the planes associated with C to the point that we are able to construct all planes of order 9 having C as a collineation group. In each case, the planes constructed are among the known ones. Thus,

0097-3165/84 $3.00

Copyright © 1984 by Academic Press, Inc.
All rights of reproduction in any form reserved.

additional planes of order 9, if they exist, have somewhat restricted collineation groups. More precisely, we show the only possible prime divisors of such a group to be 3 or 5. In addition, the 5-sylow subgroups of C are shown to have order at most 5.

2. A Reduction Lemma

We begin by narrowing the possibilities for collineations of prime order in a projective plane of order 9. In particular, we will show that the only such possible divisors are 2, 3, 5, 7, and 13. It is interesting to note that each of these primes does in fact occur in one or more of the four known planes. For example, the order of the full collineation group for the Desarguesian plane is $2^8 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ [1], while the Hughes plane has order $2^5 \cdot 3^4 \cdot 13$ [9]. Thus any further attempts to eliminate primes as possible divisors must take into account the structure of the four known planes.

The following two fundamental results, see [6], will be used in establishing Lemma 2.3.

Result 2.1. *A collineation of a finite projective plane has an equal number of fixed points and lines.*

Result 2.2. *If a nontrivial collineation has more than n fixed points, then the number of its fixed points must be $n+1$, $n+2$ or $n+\sqrt{n}+1$. Furthermore, the number of fixed points is $n+\sqrt{n}+1$ if and only if the fixed points form a subplane of order $\sqrt{n}$.*

Lemma 2.3. *If a projective plane $\mathscr{P}$ of order 9 has a collineation σ of prime order p, then p is 2, 3, 5, 7, or 13.*

Proof.

Case 1. Suppose σ has no fixed points. Then every element of $\mathscr{P}$ belongs to an orbit of size p. Since the union of all of these orbits is $\mathscr{P}$, p divides 91. Hence $p = 7$ or 13.

Case 2. We assume σ has at least one fixed point. By Result 2.1 σ has at least one fixed line, say l. Clearly the points on l must be permuted among themselves in orbits of length 1 or p. If l contains a nontrivial orbit under σ, then the length of that orbit can be no longer than 10 (the total number of points on l); thus $p = 2$, 3, 5, or 7. Suppose then that l contains only fixed points. By Result 2.2, the total number of fixed points must be either 10, 11, or 13. However, 13 fixed points corresponds to a projective subplane of order 3. This contradicts the assumption that l is a line consisting solely of fixed points. If the number of fixed points is either 10 or 11, then the number of fixed lines is 10 or 11, respectively. There exists a fixed line with at most two fixed points in either case and hence by the argument given above, p must equal 2, 3, 5, or 7.

Recently Janko and van Trug [7] have shown that only the four known planes contain collineations of order 2, thus narrowing the field to the primes 3, 5, 7, and 13.

3. Collineations of Order 13

Given a projective plane $\mathscr{P}$ of order 9, we know from the proof of Lemma 2.3:

Remark 3.1. A collineation of order 13 in $\mathscr{P}$ has no fixed elements.

Clearly then neither of the Hall planes has a collineation of order 13 as each of these planes contains a special element fixed by all collineations. Thirteen, however, does divide the order of the full collineation group of both the Desarguesian and Hughes planes. In the next four sections we will prove that there are no more planes of order 9 with this property. That is,

Lemma 3.2. *If $\mathscr{P}$ is a projective plane of order nine whose collineation group contains an element of order* 13, *then $\mathscr{P}$ is either the Desarguesian or the Hughes planes.*

4. Orbits of σ and the Matrix M

Suppose that $\mathscr{P}$ is a projective plane of order 9 whose collineation group contains a collineation σ of order 13. We know that σ has seven point orbits and seven line orbits of length 13. Label these point and line orbits $P_1, P_2, \ldots, P_7$ and $L_1, L_2, \ldots, L_7$, respectively.

We define a matrix M whose rows and columns represent point and line orbits of σ.

Definition. $M = [m_{ij}]$, $1 \leqslant i, j \leqslant 7$, where m_{ij} is the number of points of orbit P_j which lie on any given line of L_i.

M is well defined and we note that since σ establishes a one-to-one correspondence between the lines of L_i which meet any given point of P_j and the points of P_j which meet any given line of L_i, m_{ij} is also the number of lines of L_i which meet any given point of P_j.

Remark 4.1. $M = [m_{ij}]$ is a 7×7 matrix of integers possessing the following properties:

(a) $0 \leqslant m_{ij} \leqslant 4$,

(b) $\sum_{i=1}^{7} m_{ij} = \sum_{j=1}^{7} m_{ij} = 10$ for $1 \leqslant i, j \leqslant 7$,

(c) $\sum_{k=1}^{7} m_{ik} \cdot m_{jk} = \sum_{k=1}^{7} m_{ki} \cdot m_{kj} = 13$ for $1 \leqslant i \neq j \leqslant 7$.

Proof. If p is any point of P_j, then the number of lines through p of L_i is m_{ij}. On each of these lines are $(m_{ij} - 1)$ additional points of P_j. Thus $|P_j|$ is at least

$$1 + m_{ij}(m_{ij} - 1) \qquad \text{for} \quad 1 \leqslant i, j \leqslant 7. \tag{1}$$

Hence $m_{ij} \leqslant 4$, since $|P_j| = 13$.

Property (b) is clear since $\sum_{i=1}^{7} m_{ij}$ equals the total number of lines passing through a point of P_j. To establish (c), we count for fixed i and j $(1 \leqslant i \neq j \leqslant 7)$ the number of triples

$$(l_i, l_j, p) \; l_i \text{ in } L_i \text{ and } l_j \text{ in } L_j, \, p \text{ meets } l_i \text{ and } l_j$$

in two ways and obtain

$$13 \cdot 13 \cdot 1 = 13 \sum_{k=1}^{7} m_{ik} \cdot m_{jk}. \tag{2}$$

Thus

$$\sum_{k=1}^{7} m_{ik} \cdot m_{jk} = 13. \tag{3}$$

An analogous count gives

$$\sum_{k=1}^{7} m_{ki} \cdot m_{kj} = 13. \tag{4}$$

Remark 4.2. M has three types of rows and columns:

Type (1) containing two 3's, four 1's and one 0,

Type (2) containing one 3, three 2's, one 1 and two 0's,

Type (3) containing one 4 and six 1's.

Proof. For a given row or column let n_i be the number of its entries equal to i. Consider the ordered triples (x, x', y), where x and x' are distinct elements of the orbit represented by the given row or column and y is the element of $\mathscr{P}$ which is determined by x and x'. Counting these triples in two ways we find

$$13 \cdot 12 \cdot 1 = 13 \cdot n_4 \cdot 4 \cdot 3 + 13 \cdot n_3 \cdot 3 \cdot 2 + 13 \cdot n_2 \cdot 2 \cdot 1. \tag{5}$$

Thus

$$6 = 6n_4 + 3n_3 + n_2. \tag{6}$$

Furthermore by (b) of Remark 4.1 we know that

$$10 = 4n_4 + 3n_3 + 2n_2 + n_1. \tag{7}$$

Finally, since M is a 7×7 matrix, it is clear that

$$7 = n_4 + n_3 + n_2 + n_1 + n_0. \tag{8}$$

These equations imply Result 4.2.

5. Possibilities for the Matrix M

We wish to construct all matrices, up to permutation of rows and columns, having row and column types given in Remark 4.2. Possibilities for such matrices must surely exist as two of the four known planes contain collineations of order 13. Fortunately, we can identity (and eliminate) these quite early in our investigation. They correspond to matrices containing a type 3 row and column. To see this, let us suppose that M contains a type 3 row and, consequently, a type 3 column. Assume

$$r_1 = (4, 1, 1, 1, 1, 1, 1) \qquad \text{and} \qquad c_1 = r_1^{\mathrm{T}}.$$

Then

Remark 5.1. $B = P_1 \cup L_1$ represents a Baer subplane of order 3 in $\mathscr{P}$.

Proof. Any pair of points in P_1 determines a line of L_1 as all entries other than m_{11} in c_1 equal 1. Similarly any pair of lines in L_1 determines a point of P_1. Since each line of L_1 contains at most four points of P_1, P_1 contains a quadrangle. Let n be a line of $\mathscr{P}$ not contained in B. Assume that n belongs to line orbit L_i. Since $m_{i1} = 1$, every line of L_i meets P_1 in a single point. In particular, n meets B in $P_1 \cup L_1$ in a single point. Similarly every point p of $\mathscr{P}$ not in B meets B in a single line. Hence B is Baer.

However, Killgrove, Parker, and Milne [8] have shown

THEOREM. *If a plane of order nine has at least one Baer subplane, then it is one of the four known ones.*

This together with Remark 4.3 gives us

Remark 5.2. If M contains a type 3 row or column, then M cannot correspond to a new plane of order nine.

Remark 5.2 severely restricts the possibilities for M, since now we may concern ourselves solely with matrices which do not have type 3 rows or columns. Indeed, up to renaming rows and columns, there is only one such matrix. We outline a procedure for producing it in the remainder of this section.

Let r_i and c_i denote row i and column i of M, respectively.

Remark 5.3. M cannot consist solely of type 2 rows and columns.

Proof. Suppose

$$r_1 = (3, 2, 2, 2, 1, 0, 0).$$

Since the inner product of any row with r_1 is odd and since each row and column has only one 1 and one 3, either the first position of r_i is 1 or the fifth position of r_i is 3. This implies that at least one column of M is type 1.

Remark 5.4. M contains at most one row and one column of type 1.

Proof. Suppose M has at least three type 1 rows. Up to a permutation of the columns of M there are only two ways of fitting two type 1 rows together:

$$\begin{matrix} (3, 3, 1, 1, 1, 1, 0) & (3, 3, 1, 1, 1, 1, 0), \\ (1, 1, 3, 3, 1, 0, 1) & (3, 0, 1, 1, 1, 1, 3). \end{matrix}$$

A routine computation shows that the second of these arrangements must occur somewhere in the first three rows. Thus by a suitable permutation of the rows and columns of M we may assume

$$r_1 = (3, 3, 1, 1, 1, 1, 0),$$

$$r_2 = (3, 0, 1, 1, 1, 1, 3).$$

Since type 2 columns have only one 3 and one 1, columns 1, 3, 4, 5, and 6 are all type 1. Therefore M contains at least five type 1 columns. However, if a matrix M contains one type 2 column it contains at least three type 2 columns. Thus M contains only type 1 columns and consequently only type 1 rows.

We may assume that $c_1 = r_1^T$ and $c_2 = r_2^T$. This in turn allows us to assume

$$r_3 = (1, 1, 3, 3, 1, 0, 1) = c_3^T.$$

For the inner product of columns 3 and 4 to be 13 we must have $c_4 = (1, 1, 3, 0, 1, 3, 1)^T$. However, rows 3 and 5 currently inner product to eight with no way of bringing that product to 13. We have shown that M has at most two type 1 rows. Similarly M has at most two type 1 columns.

Suppose now that M contains exactly two type 1 rows, say r_1 and r_2. Notice that these rows cannot be fitted together in the second of the two ways mentioned above as that would imply M contained five type 1 columns. Thus we may assume

$$r_1 = (3, 3, 1, 1, 1, 1, 0),$$

$$r_2 = (1, 1, 3, 3, 1, 0, 1).$$

We may further assume that

$$c_5 = (1, 1, 3, 3, 1, 1, 0)^{\mathrm{T}}.$$

Since the inner product of r_1 with r_3 is odd, the sole 1 of r_3 must be in the $(3, 7)$ position. However, this forces the inner product of r_2 with r_3 to be even. In the same way we show that M has at most one type 1 column.

Remark 5.5. M has precisely one type 1 row and one type 1 column.

Proof. Assume M has exactly one type 1 row. Say

$$r_1 = (3, 3, 1, 1, 1, 1, 0).$$

Each of the remaining rows is type 2 by Remark 5.4 and these must contain in their seventh position either a 1 or a 3 since otherwise the inner product of this row with r_1 would be even. Hence c_7 is type 1.

Similarly, the existence of a type 1 column implies the existence of a type 1 row. It now follows from Remark 3.6 that M has precisely one type 1 row and column.

Finally, through a routine argument of moderate length we may show

Remark 5.6. M may be assumed to be the following matrix:

$$\begin{matrix}
3 & 2 & 1 & 2 & 2 & 0 & 0 \\
3 & 1 & 2 & 0 & 0 & 2 & 2 \\
1 & 2 & 0 & 3 & 0 & 2 & 2 \\
1 & 2 & 0 & 0 & 3 & 2 & 2 \\
0 & 3 & 3 & 1 & 1 & 1 & 1 \\
1 & 0 & 2 & 2 & 2 & 3 & 0 \\
1 & 0 & 2 & 2 & 2 & 0 & 3
\end{matrix}$$

6. Distance and Spacing

Having discovered a possibility for the matrix M, we wish to determine whether M corresponds to a projective plane of order 9. It, in fact, does not. This will become clear once we begin fitting the point and line orbits represented by M's rows and columns together. In particular, we will show there to be no consistent way for the lines of L_1, L_2, L_3, and L_4 to meet the points of P_1 and P_2.

Our task of point and line fitting will be made easier by first introducing the notions of distance and spacing in an orbit. Let a and b be distinct elements of $\mathscr{P}$ lying in the same orbit under the action of σ. Let m, n, $1 \leqslant m$, $n \leqslant 13$, be such that $\sigma^n(a) = b$ and $\sigma^m(b) = a$.

Definition. The smaller of the two integers m and n is called the *distance* between elements a and b, denoted $d(a, b)$. If $d(a, b) = 1$ we say a and b are *adjacent*.

Notice that for any pair of points a, b in P_i, $d(a, b) \leqslant 6$.

Suppose that the (i, j) entry m_{ij} of M equals 3. Then each line in L_i is incident with three points of P_j, and each point of P_j is incident with three lines of L_i. We define the idea of a spacing in such a case as follows:

Let l be any line of L_j and a, b, and c be the 3 points of P_i incident with l. Assume without loss of generality that $d(a, b) \leqslant d(b, c) \leqslant d(a, c)$.

Definition. The ordered triple

$$(d(a, b), d(b, c), d(a, c))$$

is called the *spacing* of point orbit P_i relative to line orbit L_j.

Similarly suppose p is any point of P_j and l, m and n are three lines of L_i passing through p arranged so that $d(1, m) \leqslant d(m, n) \leqslant d(1, n)$. Then

Definition. The ordered triple

$$(d(1, m), d(m, n), d(1, n))$$

is called the *spacing* of the line orbit L_i relative to the point orbit P_j.

Remark 6.1. The spacing of L_i relative to P_j equals the spacing of P_j relative to L_i. We will take advantage of this use these two notions interchangeably.

7. The Spacings of $\mathscr{P}$

Assume that the matrix M corresponds to a projective plane $\mathscr{P}$ of order nine. Consider the entry $m_{11} = 3$ in M. We study the possible spacings of P_1 relative to L_1. Let l be a line of L_1. Label the points of P_1 $p_1, p_2, ..., p_{13}$ so that σ moves them cyclically modulo 13, that is, $\sigma(p_i) = p_{i+1}$ for $1 \leqslant i \leqslant 12$ and $\sigma(p_{13}) = p_1$. Furthermore, assume that the subscripts have been chosen so that p_1 is incident with l.

If d is the smallest distance between points p_i on l, then replace σ by its dth power and relabel the points of P_1 appropriately. Relative to the new σ, l meets P_1 in the two adjacent points p_1 and p_2.

Remark 7.1. The possible spacings for P_1 relative to L_1 are $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, and $(1, 5, 6)$.

Proof. No two entries of the ordered triple could be equal: otherwise, l would intersect one of its images in at least two points. Also, since $d(a, b) = 1$, $d(a, c) = d(b, c) + 1$.

Since $m_{21} = 3$ we can examine the spacing of the points of P_1 relative to the lines of L_2. If (d_1, d_2, d_3) is the spacing of P_1 relative to L_1, then no two points of P_1 lying on a line of L_2 can be distance d_1, d_2, or d_3 apart as this would imply $L_1 \cap L_2 \neq \varnothing$. Thus for each of the four permissible spacings of P_1 relative to L_1 we have precisely one candidate for a permissible spacing of P_1 relative to L_2. These spacings are given in Table I.

Notice, however, that $(4, 5, 6)$, $(2, 3, 6)$, and $(2, 3, 4)$ could not possibly represent true spacings. Hence we have

Remark 7.2. The spacings of P_1 relative to L_1 and L_2 are $(1, 3, 4)$ and $(2, 5, 6)$, respectively.

By a dual argument we can show that the spacings of L_5 relative to P_2 and P_3 are $(1, 3, 4)$ and $(2, 5, 6)$. Unfortunately, since we have already fixed the power of σ under consideration, we do not know which spacing goes with

TABLE I

Spacing of P_1	
Relative to L_1	Relative to L_2
$(1, 2, 3)$	$(4, 5, 6)$
$(1, 3, 4)$	$(2, 5, 6)$
$(1, 4, 5)$	$(2, 3, 6)$
$(1, 5, 6)$	$(2, 3, 4)$

which point orbit. Notice, however, that if (2, 5, 6) were the spacing of L_5 relative to P_2 then no two points of P_2 on a line of L_1 could be distance 2, 5, or 6 apart. Because the spacing of P_1 relative to lines of L_1 is (1, 3, 4), these are the only candidates for the distance between two points of P_2 on a line of L_1. Thus (2, 5, 6) could not be the spacing of P_2 relative to L_5 and we have

Remark 7.3. The spacings of L_5 relative to P_2 and P_3 are (1, 3, 4) and (2, 5, 6), respectively.

We are now in a position to eliminate the matrix M thereby completing the proof of Lemma 3.2. This is accomplished by

Remark 7.4. There is no consistent way for the lines of L_1, L_2, L_3, and L_4 to meet the points of P_1 and P_2.

Proof. Recall from Remark 7.2 that any line of L_1meets the points of P_1 in three points spaced distance 1, 3, and 4 apart. In particular, by considering σ^{-1} if necessary, we may assume that line l of L_1 meets P_1 in points p_1, p_{10}, and p_{13}.

As in the case of point orbit P_1 we label the points of P_2 $q_1, q_2, \ldots, q_{13}$ so that σ moves them cyclically modulo 13 and so that line l of L_1 meets P_2 in q_1. Again from Remark 7.2, we know that a line of L_2 meets P_1 in 3 points spaced distances 2, 5, and 6. Let m be the line of L_2 meeting P_1 in p_1 and p_3. The third point of P_1 on m must be either p_8 or p_9, so we have

Case a. m meets P_1 in points p_1, p_3, and p_8

or

Case b. m meets P_1 in points p_1, p_3, and p_9.

Let us assume that we are in Case a.

Next we determine how line l of L_1 meets P_2. Recall that $m_{12} = 2$. Thus l is incident with precisely two points of P_2. Since the spacing of P_1 relative to L_1 is (1, 3, 4), the two points, a and b, of P_2 on line l of L_1 cannot be distance 1, 3, or 4 apart.

There are three cases to consider corresponding to $d(a, b) = 2$, 5, or 6. We will complete the case where $d(a, b) = 2$ and note that the remaining cases follow similarly.

Let l meet P_2 in points q_1 and q_{12}. From the matrix M we know that line m is incident with exactly one point of P_2. We determine which point of P_2 lies on m.

Note that p_1 is incident with l, $\sigma(l)$, and $\sigma^4(l)$ as well as m. If m were incident with any one of q_1, $q_2 = \sigma(q_1)$, $q_3 = \sigma^4(q_{12})$, $q_5 = \sigma^4(q_1)$, q_{12}, or $q_{13} = \sigma(q_{12})$, then m would contain two distinct points on a line of L_1. This contradicts the fact that L_1 and L_2 are distinct line orbits. Hence q_4, q_6, q_7, q_8, q_9, q_{10}, and q_{11} are the only candidates for points of P_2 incident with m.

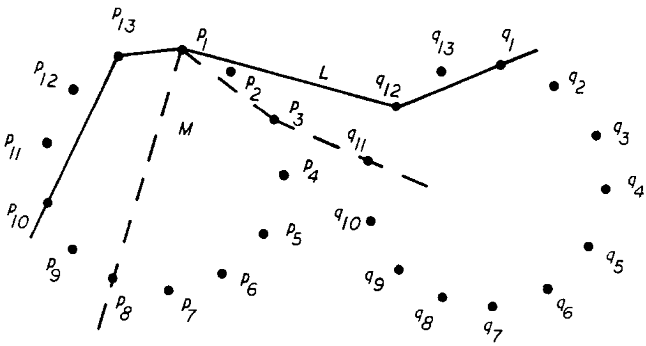

FIG. 1. Summary of incidences between points of P_1, P_2 and lines of L_1, L_2.

We can use the fact that p_1 is incident with m, $\sigma^6(m)$, and $\sigma^{-2}(m)$ to eliminate all but one of these possibilities. For example, if q_4 were incident with m, then $\sigma^{-2}(m)$ would contain both p_1 and q_2. But as we noted above $[p_1, q_2]$ is a line of L_1. Using similar arguments we can eliminate every possibility except q_{11}. Thus

Remark 7.5. m meets P_2 in q_{11}.

We summarize these results in Fig. 1.

A line of orbit L_3 or L_4 meets P_1 in exactly one point and P_2 in two points. Let r and s be lines of L_3 and L_4, respectively, passing through p_1. As both r and s contain p_1, neither line is incident with any point of P_2 lying on l, $\sigma(l)$, $\sigma^4(l)$, m, $\sigma^6(m)$, or $\sigma^{-2}(m)$. This limits the possibilities for such incidences to the points q_6, q_7, q_8, and q_{10}. Hence we have

Remark 7.6. Lines r and s meet P_2 in points q_6, q_7, q_8, and q_{10}.

Recall that the spacing of L_5 relative to P_2 is $(1, 3, 4)$, while lines of L_1 meet P_2 in a pair of points distance 2 apart. This implies that r and s each meet P_2 in two points distance either 5 or 6 apart. Note, however, that no two points of the set q_6, q_7, q_8, q_{10} are distance more than 4 apart.

Case b can be eliminated by similar arguments, completing the proof of Remark 7.4.

8. COLLINEATIONS OF ORDER 7

In this and the next three sections we consider collineations of order 7 in a projective plane of order 9. We begin by determining possible associated fixed point structures.

DEFINITION. Given a collineation σ of $\mathscr{P}$, we define $f(\sigma)$ to be the number of fixed points of σ.

Note, by Result 2.1, $f(\sigma)$ also equals the number of fixed lines of σ.

Remark 8.1. Let σ be a collineation of order 7 of $\mathscr{P}$. Then either $f(\sigma)=0$; or $f(\sigma)=7$, in which case the fixed elements form a subplane of order 2 in $\mathscr{P}$.

Proof. Each orbit of σ contains either one or seven points. If n is the number of nontrivial orbits under σ, that is, orbits of length greater than 1, then

$$7 \cdot n + f(\sigma) = 91 = 13 \cdot 7. \tag{9}$$

Thus $f(\sigma) = 0 \pmod 7$. Since $f(\sigma) \leqslant 13$ (Result 2.2), $f(\sigma)=0$ or $f(\sigma)=7$.

Suppose that σ has exactly 7 fixed points and hence exactly 7 fixed lines. Each fixed line must consist of one orbit of length 7 and three fixed points. This implies the existence of a quadrangle of fixed points. The fixed elements clearly form a closed configuration and hence a subplane of order 2.

Whitesides [11] has shown that only the Desarguesian plane contains a collineation of order 7 with no fixed points. Thus we need only concern ourselves with collineations of order 7 fixing a subplane of order 2. Since 7 does *not* divide the orders of the collineation groups of the Hall or Hughes planes of order nine, and since the Desarguesian plane has no subplanes of order 2, no known plane of order 9 contains a collineation of order 7 fixing a subplane of order 2 pointwise. We will show that *no* plane of order 9 could contain such a collineation. Consequently,

LEMMA 8.2. *If a projective plane $\mathscr{P}$ of order 9 has a collineation of order 7, then $\mathscr{P}$ is Desarguesian.*

9. THE MATRIX M REVISITED

Suppose $\mathscr{P}$ is a projective plane of order 9 and σ is a collineation of $\mathscr{P}$ with order 7 fixing a subplane of order 2 pointwise. Let T_0 be the fixed elements of σ.

Each line l of T_0 contains precisely three points of T_0. The remaining seven points of l must form an orbit of σ. Hence there are seven point orbits of σ associated with the lines of T_0, one orbit on each fixed line of σ. Similarly there are seven line orbits of σ associated with the points of T_0, one orbit through each fixed point of σ. Denote these point and line orbits $P_1, P_2, \ldots, P_7$ and $L_1, L_2, \ldots, L_7$, respectively. Let $P_8, P_9, \ldots, P_{12}$ be the remaining nontrivial point orbits and $L_8, L_9, \ldots, L_{12}$ be the remaining nontrivial line orbits.

As in Section 4, we fill out the incidence matrix M whose rows and columns represent the nontrivial point and line orbits of σ. More precisely, $M = [m_{ij}]$, $1 \leqslant i, j \leqslant 12$, where m_{ij} is the number of points of orbit P_j which lie on any given line of L_i. Recall that m_{ij} may also be interpreted as the number of lines of L_i which meet any given point of P_j. The matrix M is divided into blocks, as shown below, for easier reference.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
L_1												
L_2												
L_3												
L_4				BLOCK I						BLOCK II		
L_5												
L_6												
L_7												
L_8												
L_9												
L_{10}				BLOCK III						BLOCK IV		
L_{11}												
L_{12}												

Consider a line l of L_i, $1 \leqslant i \leqslant 7$. By definition l passes through a point x of T_0. Precisely three lines of T meet l in x. If two of the four remaining lines of T_0 met l in a point y, y and hence l would be fixed by σ. Thus the remaining four lines of T_0 meet l in four distinct points. This implies that l contains precisely four points from orbits $P_1, P_2, \ldots, P_7$ with no more than one point from each orbit. The five points unaccounted for on l must come from orbits $P_8, P_9, \ldots, P_{12}$. Furthermore at most one point of P_j, $8 \leqslant j \leqslant 12$, lies on l since otherwise l would intersect one of its images in two points. Hence

Remark 9.1. Each line of orbit L_i, $1 \leqslant i \leqslant 7$, contains one point of T_0, 1 point from each of four distinct orbits P_k, P_l, P_m, P_n, $1 \leqslant k, l, m, n \leqslant 7$, and one point from each of the last five orbits $P_8, P_9, \ldots, P_{12}$.

The dual argument yields

Remark 9.2. A point x of point orbit P_j, $1 \leqslant j \leqslant 7$, lies on one line of T_0, one line from each of four distinct orbits L_k, L_l, L_m, L_n, $1 \leqslant k, l, m, n \leqslant 7$, and one line from each of the last five orbits $L_8, L_9, \ldots, L_{12}$.

Together Remarks 9.1 and 9.2 imply

Remark 9.3. Blocks II and III of M consist entirely of 1's.

Next we consider Block IV. Let n_i be the number of entries equal to i in some given row of Block IV. We count ordered triples (l, l', p), where l, l' are in L_i, $8 \leqslant i \leqslant 12$, and $p = l \cap l'$ in two ways to obtain

$$7 \cdot 6 \cdot 1 = 7 \cdot (6 \cdot n_3 + 2 \cdot n_2) + \text{contributions not in IV.} \tag{10}$$

Note, however, that since all the entries in Block III equal 1, there are no contributions from points in $P_1, P_2, \ldots, P_7$. Also, there are no contributions from points of T_0. Since the number of points unaccounted for on a line of L_i, $8 \leqslant i \leqslant 12$, is three, we have

$$3 \cdot n + 2 \cdot n_2 + n_1 = 3. \tag{11}$$

These two equations imply $n_3 = 1$, $n_2 = 0$, and $n_1 = 0$. Since the dual argument shows that each column of Block IV has one 3 and four 0's, we can label orbits so that

Remark 9.4. Block IV may be assumed to be

$$\begin{matrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{matrix}$$

Finally we consider Block I.

Remark 9.5. Block I is a 7×7 0, 1 matrix possessing the following properties:

(a) $\sum_{i=1}^{7} m_{ij} = \sum_{j=1}^{7} m_{ij} = 4$ for $1 \leqslant i, j \leqslant 7$.

(b) $\sum_{k=1}^{7} m_{ik} \cdot m_{jk} = \sum_{k=1}^{7} m_{ki} \cdot m_{kj} = 2$ for $1 \leqslant i \neq j \leqslant 7$.

Proof. Property (a) is immediate from Remarks 9.1 and 9.2. To establish (b), we count for fixed i and j $(1 \leqslant i \neq j \leqslant 7)$ the number of triples (l_i, l_j, p), l_i in L_i, l_j in L_j, and $p = l_i \cap l_j$, in two ways to obtain

$$7 \cdot 7 \cdot 1 = 7 \cdot \sum_{k=1}^{7} m_{ik} \cdot m_{jk} + \text{contributions not in I.} \tag{12}$$

Each line of L_i contains six points not accounted for in Block I. One point comes from T_0 and is incident with no lines of L_j. Each of the remaining points lies in one of the orbits $P_8, P_9, \ldots, P_{12}$ and in incident with exactly one line of L_j. Hence

$$7 \cdot 7 = 7 \cdot \sum_{k=1}^{7} m_{ik} \cdot m_{jk} + 7 \cdot 5. \tag{13}$$

It is now easy to show

Remark 9.6. The only possibility, up to naming orbits, for Block I is

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Together Remarks 9.3, 9.4, and 9.6 show

Remark 9.7. M may be assumed to be the following matrix:

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
L_1	1	1	1	1	0	0	0	1	1	1	1	1
L_2	1	1	0	0	1	1	0	1	1	1	1	1
L_3	1	0	1	0	0	1	1	1	1	1	1	1
L_4	1	0	0	1	1	0	1	1	1	1	1	1
L_5	0	1	1	0	1	0	1	1	1	1	1	1
L_6	0	1	0	1	0	1	1	1	1	1	1	1
L_7	0	0	1	1	1	1	0	1	1	1	1	1
L_8	1	1	1	1	1	1	1	3	0	0	0	0
L_9	1	1	1	1	1	1	1	0	3	0	0	0
L_{10}	1	1	1	1	1	1	1	0	0	3	0	0
L_{11}	1	1	1	1	1	1	1	0	0	0	3	0
L_{12}	1	1	1	1	1	1	1	0	0	0	0	3

10. Construction of $\mathscr{P}$

Again, as in Section 7, we wish to determine whether the matrix M corresponds to a projective plane of order 9. As before, we use the completed matrix M to help us construct such a plane, should it exist, or to reach a contradiction if it does not. It does not. We prove this in the remainder of this section.

Assume $\mathscr{P}$ is a projective plane of order 9 having a collineation of order 7, σ, fixing a Fano subplane and corresponding orbit matrix M.

Remark 10.1. The 3's appearing in Block IV represent projective subplanes of order 2 in $\mathscr{P}$.

Proof. Consider $P_i \cup L_i$ for $8 \leqslant i \leqslant 12$. Every pair of points in P_i determines a line of L_i as all other entries in column i equal 1, and since no point of P_i for $8 \leqslant i \leqslant 12$ lies on any fixed line. Similarly every pair of lines in L_i determines a point of P_i. Finally, as each line of L_i contains exactly three points of P_i, P_i contains a quadrangle.

DEFINITION. Let $T_i = P_{i+7} \cup L_{i+7}$ for $1 \leqslant i \leqslant 5$.

Let $p_1^{(0)}, p_2^{(0)},..., p_7^{(0)}$ and $l_1^{(0)}, l_2^{(0)},..., l_7^{(0)}$ be the points and lines of T_0. Label the points and lines of T_i, $1 \leqslant i \leqslant 5$, $p_1^{(i)}, p_2^{(i)},...,p_7^{(i)}$, and $l_1^{(i)}, l_2^{(i)},..., l_7^{(i)}$ so that σ permutes them cyclically modulo 7. Choose the labeling so that $p_1^{(i)} = l_1^{(i)} \cap l_2^{(i)}$.

For a fixed k, $1 \leqslant k \leqslant 5$, define the incidence matrix $M_k = [m_{ij}]$, where $m_{ij} = 1$ if $l_i^{(k)}$ meets $p_j^{(k)}$ and 0, otherwise.

Remark 10.2. M_k, $1 \leqslant k \leqslant 5$, is equal to one of the following two matrices:

$$
\begin{matrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1
\end{matrix}
\qquad
\begin{matrix}
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1
\end{matrix}
$$

By using σ^{-1} as our collineation instead of σ, if necessary, we may assume that the incidence matrix M_1 of T_1 is the first matrix given in Remark 10.2.

We use the labelings for the lines of T_0 and T_1 to coordinatize the points of the first seven points orbits.

DEFINITION. Let $p_{ij} = l_i^{(0)} \cap l_j^{(1)}$, $1 \leqslant i, j \leqslant 7$.

From row 8 of matrix M we see that every point of $\bigcup_{k=1}^{7} p_k$ lies on precisely one line of T_1. By construction the same is true for the points of $\bigcup_{k=1}^{7} p_k$ and lines of T_0. This implies that the definition is well defined and that every point of the first seven line orbits has been included in the labeling. Similarly

DEFINITION. Let $l_{ij} = [p_i^{(0)}, p_j^{(1)}]$, $1 \leqslant i, j \leqslant 7$.

The next step in our construction of $\mathscr{P}$ is to define a matrix N which stores the incidences of lines of T_i, $1 \leqslant i \leqslant 5$, with the points of $\bigcup_{k=1}^{7} p_k$.

DEFINITION. Let $N = [n_{ij}]$, $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 7$, where n_{ij} is the subscript of the line of T_i meeting points p_{j1}. That is, p_{j1} lies on the line $l_{n_{ij}}^{(i)}$ of T_i.

Since σ moves the points of P_i and the lines of L_j cyclically, an entry n_{ij} of the matrix N provides information on all incidences of points of P_i with lines of L_j.

Let r_i, c_j for $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 7$, be the ith row and jth column of N, respectively. By construction we have

Remark 10.3. $r_1 = (1, 1, 1, 1, 1, 1, 1)$.

Relabel, if necessary, the lines of T_i, $2 \leqslant i \leqslant 5$, so that $l_1^{(i)}$ passes through p_{11}. Also, if necessary, relabel the lines $l_2^{(0)}, l_3^{(0)}, \ldots, l_7^{(0)}$ of T_0 so that $l_j^{(2)}$ meets $l_1^{(1)}$ in p_{j1} for $1 \leqslant j \leqslant 7$. Then

Remark 10.4. $c_1 = (1, 1, 1, 1, 1)^{\mathrm{T}}$ and $r_2 = (1, 2, 3, 4, 5, 6, 7)$.

Finally, assume that the indices of subplanes T_3, T_4, and T_5 have been labeled so that $n_{32} < n_{42} < n_{52}$. There are then ten possibilities for c_2 in N.

Remark 10.5. We may assume that c_2^{T} equals one of the following:

$(1, 2, 5, 6, 7)$, $(1, 2, 4, 5, 6)$, $(1, 2, 4, 5, 7)$, $(1, 2, 4, 6, 7)$, $(1, 2, 3, 4, 5)$,

$(1, 2, 3, 4, 6)$, $(1, 2, 3, 4, 7)$, $(1, 2, 3, 5, 6)$, $(1, 2, 3, 5, 7)$, $(1, 2, 3, 6, 7)$.

Having chosen c_2 from the list we need to complete the matrix N. This task will be made easier by observing

Remark 10.6. The 4×6 matrix obtained from N by deleting column 1 and row 1 is a Latin rectangle with integer entries between 2 and 7.

Proof. Suppose $n_{ij} = k = n_{ij'}$, for $i, j, j' \neq 1$. Then $l_k^{(i)}$ is incident with both p_{j1} and $p_{j'1}$. However, p_{j1} and $p_{j'1}$ are already incident with $l_1^{(1)}$. This contradicts the fact that $i \neq 1$. Similarly $n_{ij} = n_{i'j}$ or $n_{ij} = 1$ leads to a contradiction.

Using Remark 10.6, we program a simple backtracking algorithm to construct all possibilities for the matrix N. There are 6552 distinct possibilities for each choice of c_2. Having constructed a possibility for the matrix N, we know all incidences between points of the first seven nontrivial points orbits $\bigcup_{k=1}^{7} P_k$ and the lines of the subplanes $T_0, T_1, \ldots, T_5$. This information may now be used in an attempt to place the points ot $\bigcup_{k=1}^{7} P_k$ on the lines of $\bigcup_{k=1}^{7} L_k$. Note that since we know the action of σ on the first seven point and line orbits, it is sufficient to determine which points of $\bigcup_{k=1}^{7} P_k$ are incident with each of the lines $l_{11}, l_{21}, \ldots, l_{71}$.

We will show in the next two sections that for no choice of the matrix N is it possible to establish these incidences in a consistent fashion. This clearly suffices to prove Lemma 4.1.

11. Points and Lines from the First Seven Orbits

Recall from the matrix M that each line of $\bigcup_{k=1}^{7} L_k$ contains four points of $\bigcup_{k=1}^{7} P_k$ with no more than one point from each orbit P_k $(1 \leqslant k \leqslant 7)$.

Remark 11.1. The line $l_{i1} = [p_i^{(0)}, p_1^{(1)}]$, $1 \leqslant i \leqslant 7$, contains four points p_{a3}, p_{b5}, p_{c6}, and p_{d7} of $\bigcup_{k=1}^{7} P_k$ where a, b, c, and d are distinct integers between 1 and 7.

Proof. By Remark 10.1, $p_1^{(1)}$ is incident with lines $l_1^{(1)}$, $l_2^{(1)}$, and $l_4^{(1)}$ of T_1. Note that $l_k^{(1)}$ contains all seven points of $\bigcup_{k=1}^{7} P_k$ whose second subscript equals k. Furthermore, $l_k^{(0)}$ contains all points of $\bigcup_{k=1}^{7} P_k$ whose first subscript equals k. Thus no line other than those of T_0 contains two points of $\bigcup_{k=1}^{7} P_k$ with the same first subscript.

Remark 11.2. Each point p_{ij}, $1 \leqslant i \leqslant 7$, $j = 3, 5, 6$, or 7, occurs on precisely one line of $l_{11}, l_{21}, \ldots, l_{71}$.

Proof. The lines $l_{11}, l_{21}, \ldots, l_{71}$ are all incident with $p_1^{(1)}$ and hence no two of them are incident with the same point of $\bigcup_{k=1}^{7} P_k$. Since each of these lines contains four points of $\bigcup_{k=1}^{7} P_k$, the total number of points of $\bigcup_{k=1}^{7} P_k$ on these lines is 28.

By Remark 11.2 each of the points p_{13}, p_{15}, p_{17} must lie on a line of $l_{11}, l_{21}, \ldots, l_{71}$ and no two points lie on the same line. Assume p_{13}, p_{15}, p_{16}, and p_{17} lie on l_{a1}, l_{b1}, l_{c1}, and l_{d1}, respectively. For each possibility for the matrix N we attempt to fill out the remaining points p_{ij}, $2 \leqslant i \leqslant 7, j = 3, 5, 6$, and 7, on the four lines l_{a1}, l_{b1}, l_{c1}, and l_{d1}. Often this will not be possible since pairs of points set on one of these lines may already determine a line of T_i, $0 \leqslant i \leqslant 5$. Whenever we are unable to complete one of the lines l_{a1}, l_{b1}, l_{c1}, or l_{d1} for a particular matrix, we reject that matrix as a possibility for N. In this way we are able to eliminate most, but not all, of the possibilities for N.

Summarizing, we find that all the points and lines of $\mathscr{P}$ have been labeled. Relative to this labeling, we have constructed candidates for a matrix N storing all incidences of lines from the subplanes T_i, $1 \leqslant i \leqslant 5$, with points of $\bigcup_{i=1}^{7} P_i$. Finally, we have started fitting points of $\bigcup_{i=1}^{7} P_i$ onto the lines of $\bigcup_{i=1}^{7} L_i$ taking care not to put two points on the same line if they already determine a line of T_i, $1 \leqslant i \leqslant 5$. We have completed possibilities for these fittings corresponding to the four lines l_{a1}, l_{b1}, l_{c1}, and l_{d1} containing P_{13}, P_{15}, P_{16}, and P_{17}, respectively (managing to eliminate most of the possibilities for the incidence matrix N in the process). Recall that this yields all incidences for the line orbits L_{a1}, L_{b1}, L_c, and L_d with points of $\bigcup_{i=1}^{7} P_i$.

None of the possibilities for the constructed fittings are internally consistent as we now show.

By definition, each of the lines l_{a1}, l_{b1}, l_{c1}, and l_{d1} contains a unique point of T_0. Since a point of $\bigcup_{k=1}^{7} P_k$ has first index equal to the index of the unique line of T_0 containing it, we may use the indices of these points to determine the lines of T_0 intersecting l_{k1}, $k = a, b, c$, or d. If the result of our construction is to be a plane, it is important that points and lines of T_0 meet l_{a1}, l_{b1}, l_{c1}, and l_{d1} in a fashion which is consistent with the incident structure of a Fano plane.

Remark 11.3. Precisely two points of $\bigcup_{k=1}^{7} P_k$ with first index equal to h lie on the lines l_{a1}, l_{b1}, l_{c1}, and l_{d1} for each $h = 2, 3, \ldots, 7$.

Proof. Any two lines of T_0 miss precisely two points of T_0 simultaneously. In particular, the lines $l_1^{(0)}$ and $l_h^{(0)}$ miss two points $p_x^{(0)}$ and $p_y^{(0)}$ of T_0. These lines intersect each of the lines l_{x1}, l_{y1} in two points of $\bigcup_{k=1}^{7} P_k$ whose first indices equal 1 and h, respectively. This implies that l_{x1} and l_{y1} are two of the four lines given above and proves the remark.

Finally, we check that the incidences between points of $\bigcup_{k=1}^{7} P_k$ and the twenty-eight lines of $\bigcup_{k=1}^{7} L_k$ defined above are consistent. That is, each pair of points of $\bigcup_{k=1}^{7} P_k$ determines at most one line of $\bigcup_{k=1}^{7} L_k$.

Remark 4.17. Let m and n be two of the four lines l_{a1}, l_{b1}, l_{c1}, and l_{d1} given. Suppose p_{iw}, p_{jx} are incident with m and p_{iy}, p_{jz} are incident with n for $1 \leqslant i \neq j \leqslant 7$, w, x, y, z belonging to the set $\{3, 5, 6, 7\}$. There does not exist g, $1 \leqslant g \leqslant 7$, such that

$$w + g = y \qquad (\text{mod } 7), \tag{14}$$

$$x + g = z \qquad (\text{mod } 7). \tag{15}$$

Proof. Suppose, to the contrary, that there exists an integer g with the property given. For such a g we have

$$\sigma^g(p_{iw}) = p_{iy'}, \tag{16}$$

$$\sigma^g(p_{jx}) = p_{jz}. \tag{17}$$

Thus

$$\begin{aligned}\sigma^g(m) &= \sigma^g([p_{iw}, p_{jx}]) \\ &= [\sigma^g(p_{iw}), \sigma^g(p_{jx})] \\ &= [p_{iy}, p_{jz}] \\ &= n.\end{aligned} \tag{18}$$

However, this contradicts the fact that m and n belong to distinct line orbits of σ.

We examine each of the remaining condidates for the matrix N together with all possibilities for incidences of points from $\bigcup_{k=1}^{7} P_k$ with lines l_{a1}, l_{b1}, l_{c1}, and l_{d1}. In every case, either the incidence structure is not consistent with the structure of T_0, (Remark 11.3 fails) or the incidences are *not* internally consistent (Remark 11.4 fails). Thus

Remark 11.5. Given any possibility for the matrix N as defined, it is not possible to place the points of $\bigcup_{k=1}^{7} P_k$ on the lines of $\bigcup_{k=1}^{7} L_k$ in a consistent fashion.

This proves Lemma 8.2.

12. Collineations of Order 5

It has not yet been possible to rule out 5 as a collineation of an unknown plane of order 9. We can, however, limit the size of the 5-sylow subgroups of a collineation group associated with any projective plane of order 9. Such groups have order at most 5.

Recall $f(\sigma)$ equals the number of fixed points of the collineation σ.

Remark 12.1. A collineation of order 5 in $\mathscr{P}$ has one fixed point and one fixed line not incident with each other.

Proof. Each orbit of σ has length 1 or 5. If n is the number of orbits of length 5, then

$$5 \cdot n + f(\sigma) = 91. \tag{19}$$

Hence $f(\sigma) = 1 \pmod 5$. By Result 2.2, $f(\sigma) = 1$, 6, or 11.

Case i. $f(\sigma) = 1$.

Here σ fixes precisely one point and one line. The fixed line clearly must contain two orbits of length 5. Hence the fixed point and line cannot be incident with each other.

Case ii. $f(\sigma)=6$.

Each fixed line contains at least one fixed point, namely its intersection with another fixed line. Let k be a fixed line of σ. Since not all of the points of k are fixed, k must contain at least one point orbit of length 5. However, among the remaining five points on k there is at least one which is fixed. Hence these points cannot form a second orbit of length 5 on k. Thus k consists of one orbit of length 5 and five fixed points. If p is the remaining fixed point, p together with any fixed point of k determines a fixed line containing precisely two fixed points. This, however, is clearly impossible.

Case iii. $f(\sigma)=11$.

It is easily seen that the points of a closed configuration either: are all incident with a single line; are all except one incident with a single line; or form a projective plane. For $f(\sigma)=11$ only the second case is a possibility. Let k be the line consisting solely of fixed points and let p be the unique fixed point not incident with k. If q is any point of k, $[p,q]$ is a fixed line containing precisely two fixed points. This contradicts the fact that nontrivial orbits of σ have length 5.

LEMMA 12.2. *The full collineation group of $\mathscr{P}$ has no subgroup of order 5^2.*

Proof. Suppose to the contrary: that is, let S be a collineation group of order 5^2. Recall that any group of order p^2, where p is a prime, is abelian.

If S were cyclic, say $S=(\omega)$, then the least number of points of $\mathscr{P}$ in orbits of length less than 25 would be sixteen. Then ω^5 would be a collineation or order 5 fixing at least sixteen points, contradicting Result 12.1. Hence S is elementary abelian, say $G=(\sigma,\omega)$, where σ and ω are both collineations of order 5.

Since S is abelian, σ permutes the orbits of ω. In particular, the single fixed point and line of ω must likewise be fixed by σ. Let p be the point and k be the line fixed by both σ and ω. The two ω-orbits of length 5 on k must be fixed as sets or interchanged by σ. Clearly they must be fixed. Then, for some i, $\sigma\omega^i$ fixes two points of the plane. This again contradicts Result 12.1, as $\sigma\omega^i$ has order 5.

With this result, we have established

THEOREM 12.3. *If $\mathscr{P}$ is a projective plane of order 9 other than one of the four known ones, then the full collineation group of p is $3^a\cdot 5^b$, where a is a nonnegative integer and $b=0, 1$.*

ACKNOWLEDGMENTS

These results were contained in the author's PhD. thesis. I wish to acknowledge my deep gratitude to my thesis advisor, S. H. Whitesides for her patience and wise counsel. I would also like to thank Tom Whitesides for his fine computer programs verifying much of the work of this paper.

REFERENCES

1. H. P. DEMBOWSKI, "Finite Geometries," Ergebnisse der Mathematik und ihren Grenzgebiele 44, Springer-Verlag, New York, 1968.
2. J. DENES AND A. D. KEEDWELL, "Latin Squares and Their Applications," Academic Press, New York, 1974.
3. M. HALL, JR., Uniqueness of the projective plane with 57 points, *Proc. Amer. Math. Soc.* **4** (1953), 912–916.
4. M. HALL, JR., J. D. SWIFT, AND R. J. WALKER, Uniqueness of the projective plane of order eight, *Math. Tables Aids Comput.* **10** (1956), 186–194.
5. M. HALL, JR., J. D. SWIFT, AND R. KILLGROVE, On projective planes of order nine, *Math. Tables Aids Comput.* **13** (1959), 233–246.
6. D. HUGHES AND F. PIPER, "Projective Planes," Graduate Texts in Mathematics, Springer-Verlag, New York, 1973.
7. Z. JANKO AND T. VAN TRUNG, The classification of projective plane of order 9 which possess an involution, *J. Combin. Theory Ser. A* **33** (1982), No. 1, 65–75.
8. R. B. KILLGROVE, E. T. PARKER, AND ED MILNE, Low order projective planes, *in* "Proceeding of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing," pp. 365–390, 1976.
9. T. G. ROOM AND P. B. KIRKPATRICK, "Miniquaternian Geometry," Cambridge Univ. Press, London, 1971.
10. R. SHULL, "Collineations of Projective Planes of Order 9," PhD. thesis, Dartmouth College, Hanover, N.H., June 1980.
11. S. H. WHITESIDES, No new planes of order nine with collineations of order seven having no fixed points, to appear.